CSIDH: a post-quantum drop-in replacement for (EC)DH

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ECC Autumn School, Osaka, 17-18 November 2018


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Traditional Diffie-Hellman key exchange

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For simplicity, for a finite group $(G, *)$ and $x \in \mathbb{Z}$, we'll write g^x for $\underbrace{g * \cdots * g}_{x \text{ times}}$.

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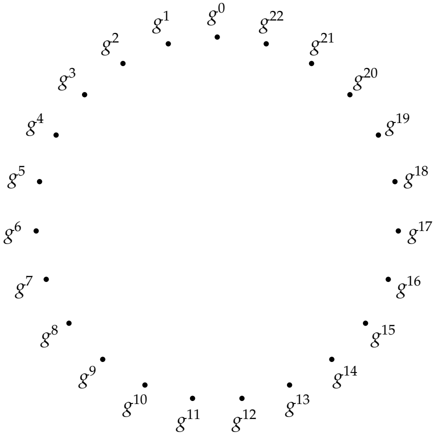
- Computing g^a given g and a should be easy (i.e. fast).

Square-and-multiply

Computing g^a: an example. Suppose $|G| = 23$ and that Alice computes g^{13}.



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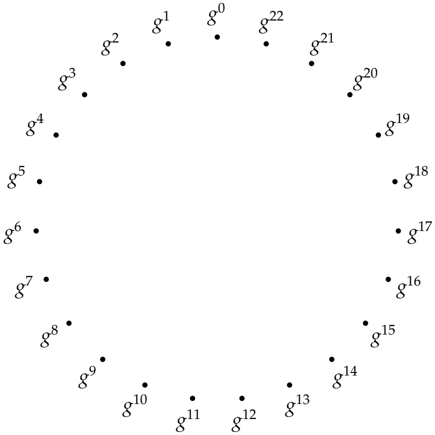

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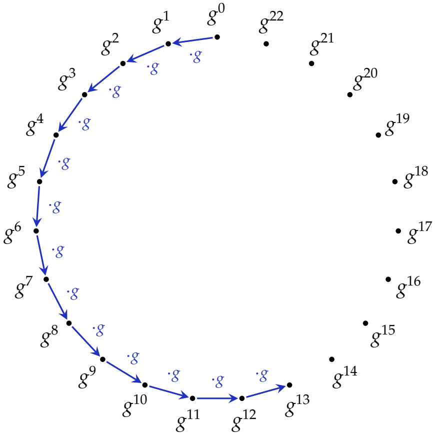

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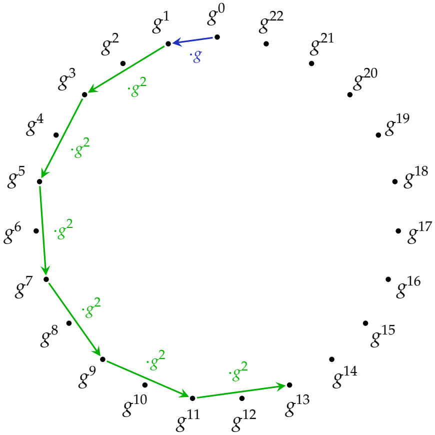

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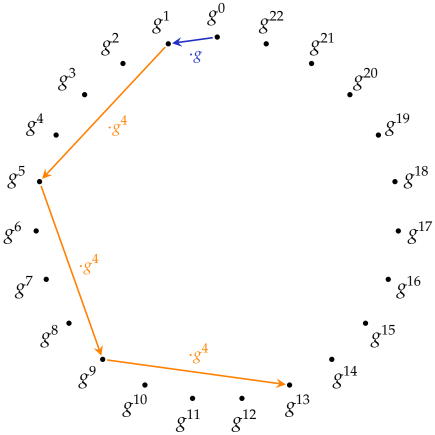

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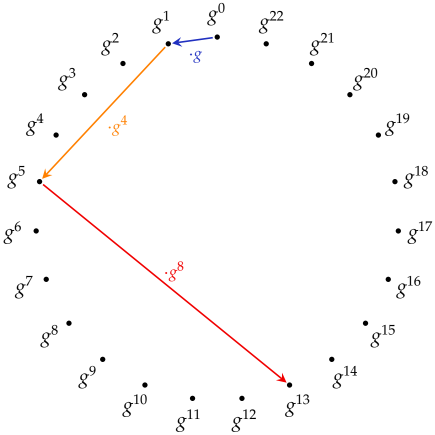

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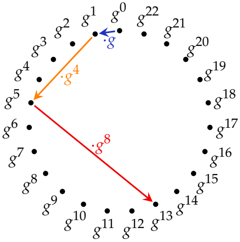

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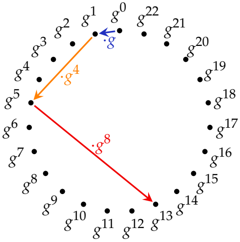

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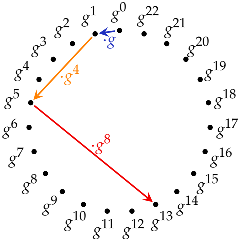

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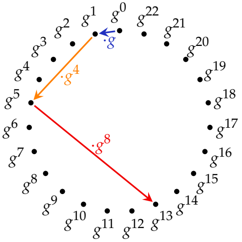

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- Exercise: prove that, for any cyclic group G of size n, if $g \in G$ and $a \in \mathbb{Z}$, Alice can compute g^a in $\leq \log_2(n)$ (multiplication) steps. (In polynomial time).

[1] a smart attacker like Mehdi can often exploit the structure of the specific group to do better than this (but even Mehdi can't manage polynomial time)

Quantum revolution

Let G be a finite group, let $g \in G$ and let $x \in \mathbb{Z}$. As before, define g^x by

$$\begin{array}{ccc} \mathbb{Z} \times G & \to & G \\ (x, g) & \mapsto & g^x := \underbrace{g * \cdots * g}_{x \text{ times}}. \end{array}$$

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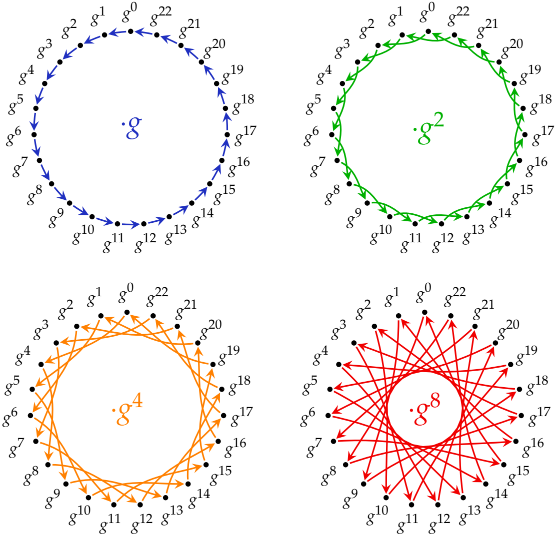
$\rightsquigarrow$ Idea:

Replace the map $\mathbb{Z} \times G \to G$ by a group action of a group H on a set S:

$$H \times S \to S.$$

What do we keep from traditional (EC)DH?



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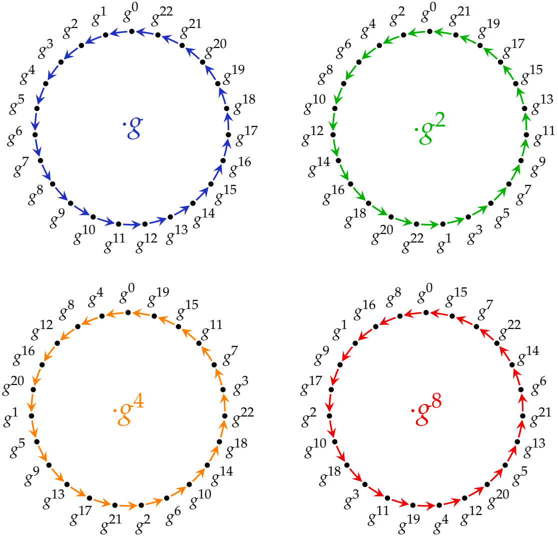

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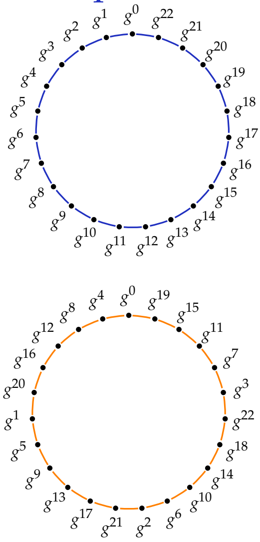

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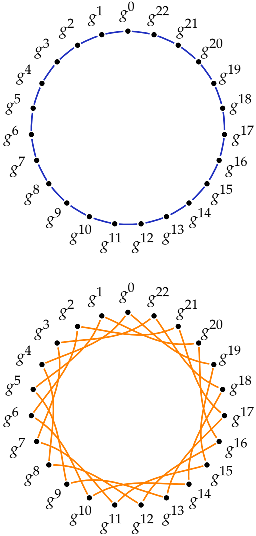

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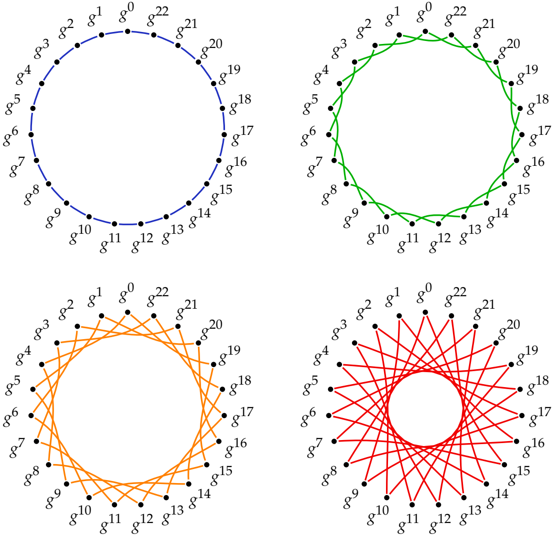

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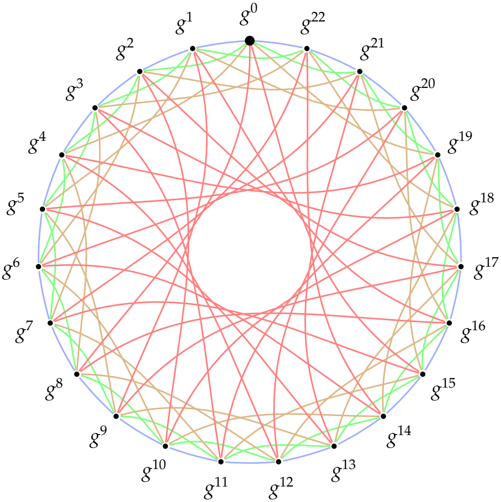

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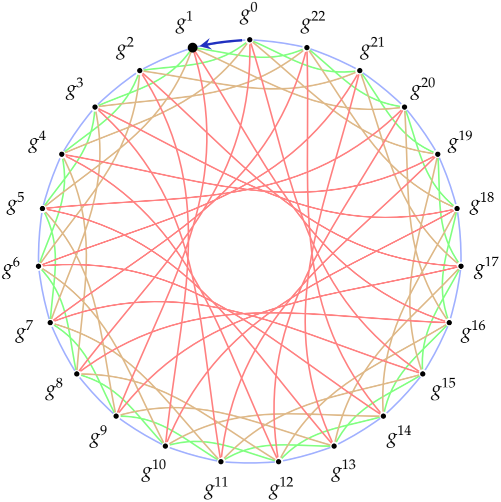

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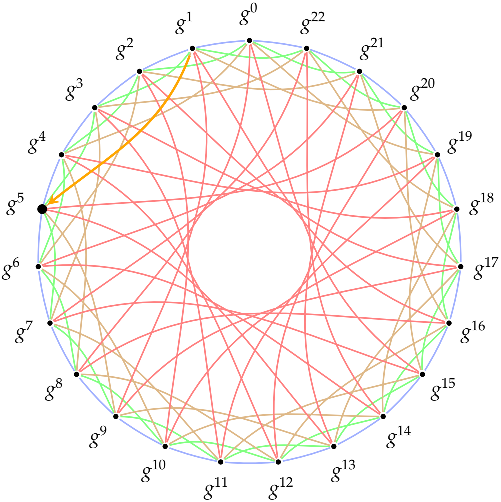

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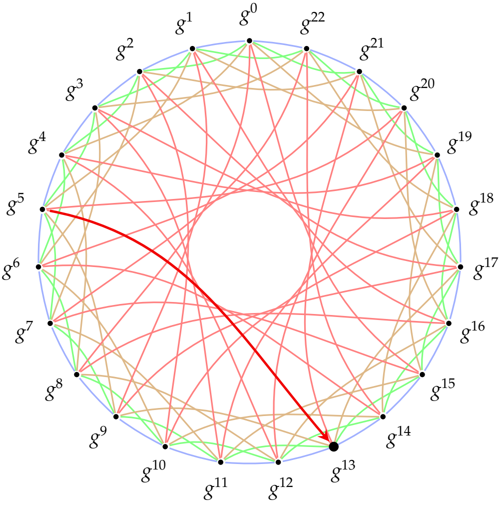

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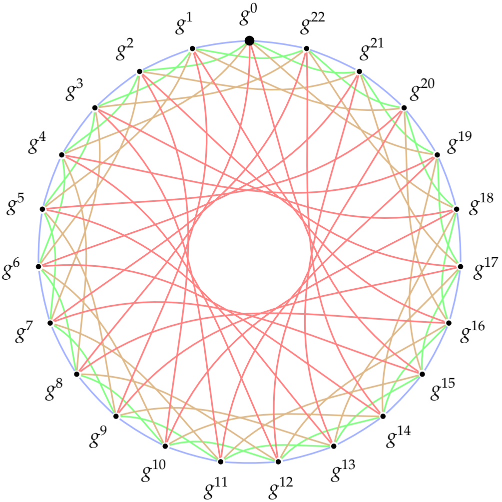

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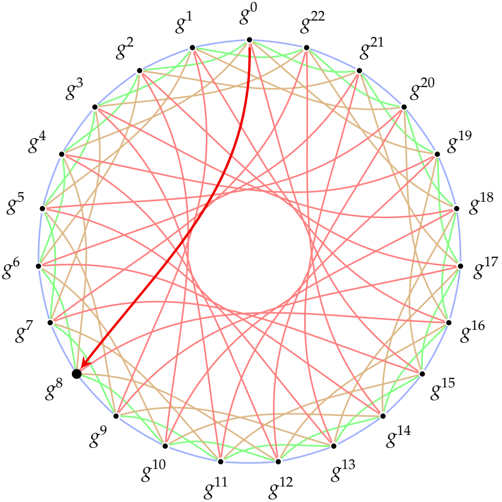

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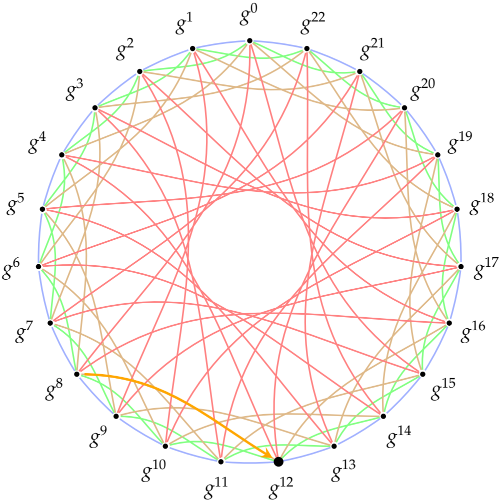

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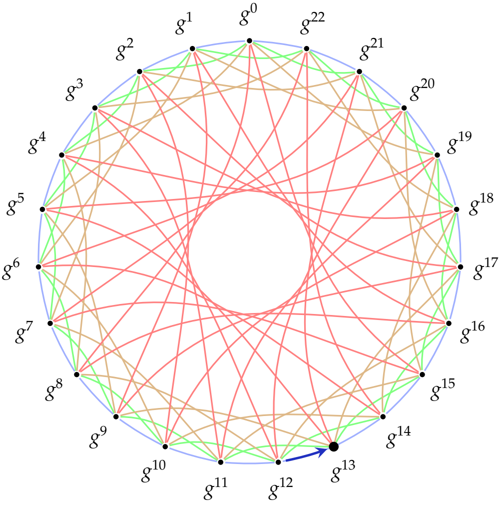

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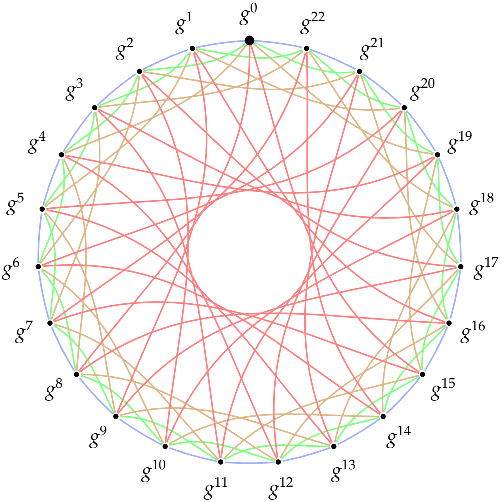

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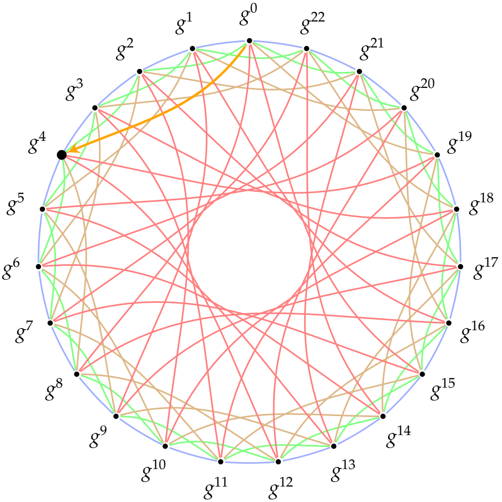

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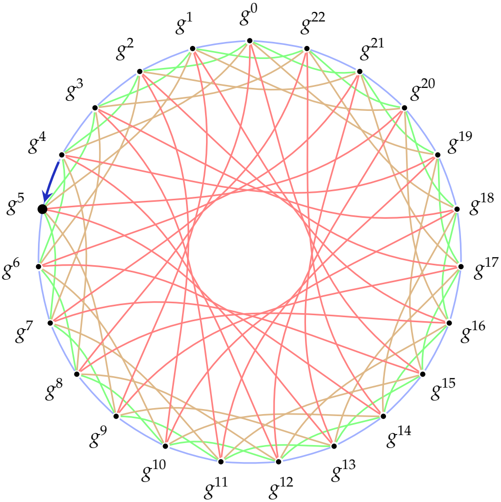

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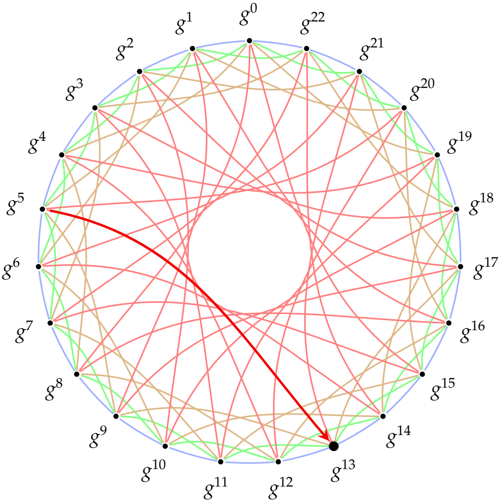

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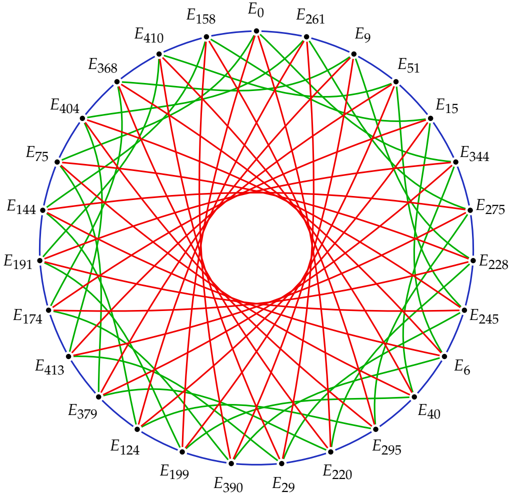
Graphs of elliptic curves



CSIDH: Nodes are now elliptic curves and edges are isogenies.

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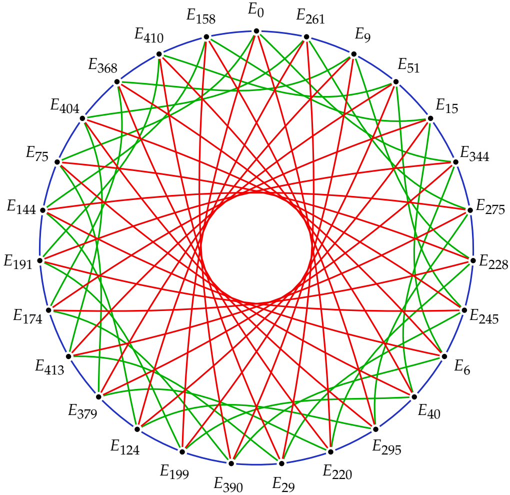

Nodes: Supersingular elliptic curves $E_A : y^2 = x^3 + Ax^2 + x$ over $\mathbb{F}_{419}$.
Edges: 3-, 5-, and 7-isogenies (more details to come).

Diffie-Hellman on 'nice' graphs

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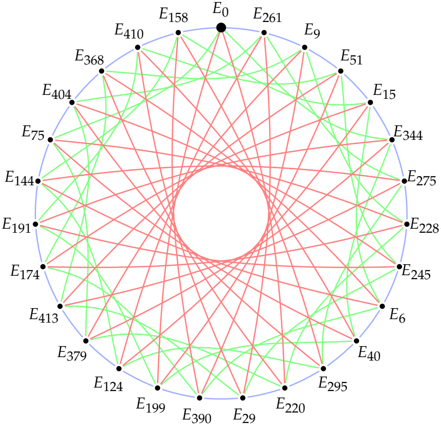
$a = [+, -, +, -]$



Bob

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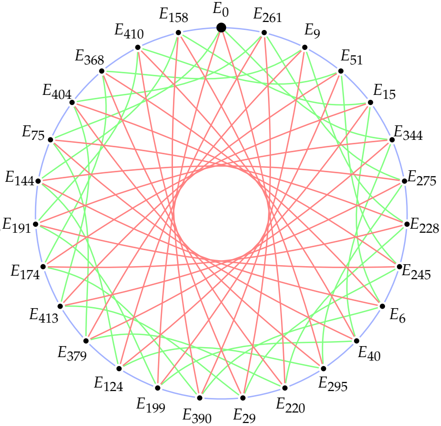

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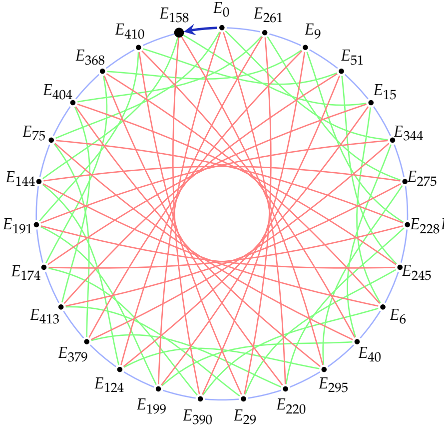

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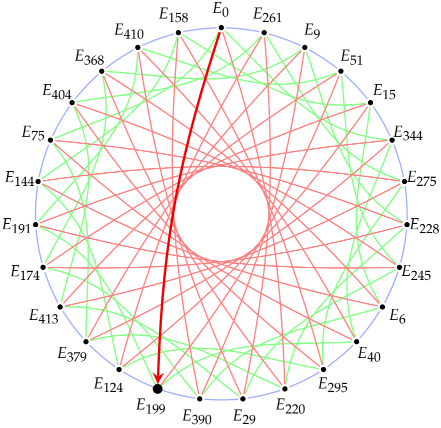

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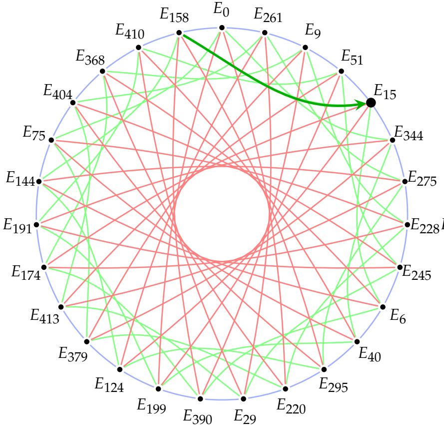

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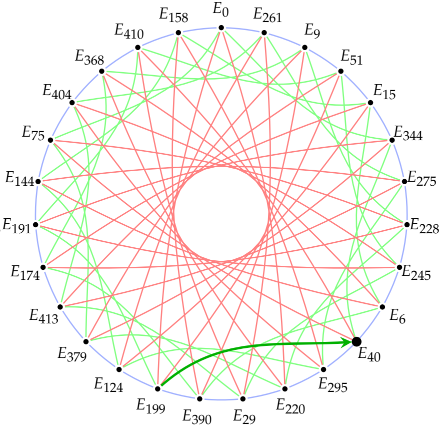

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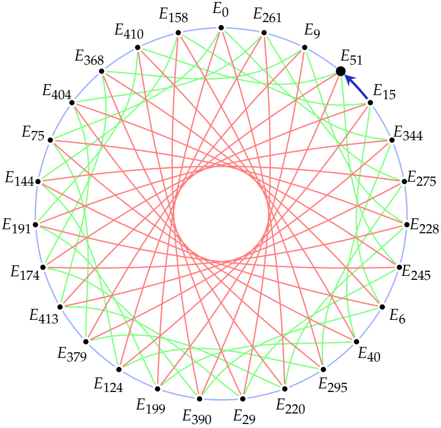

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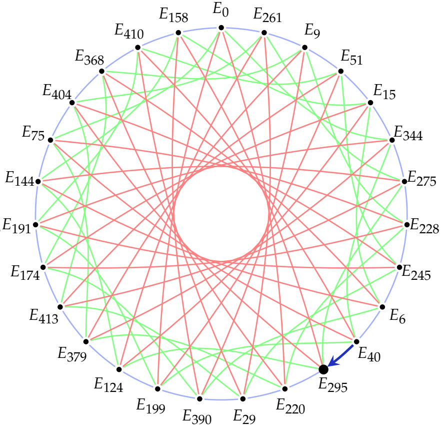

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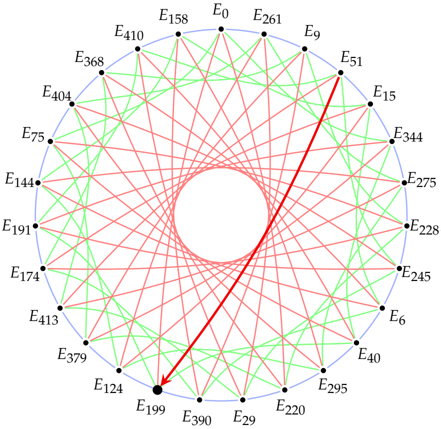

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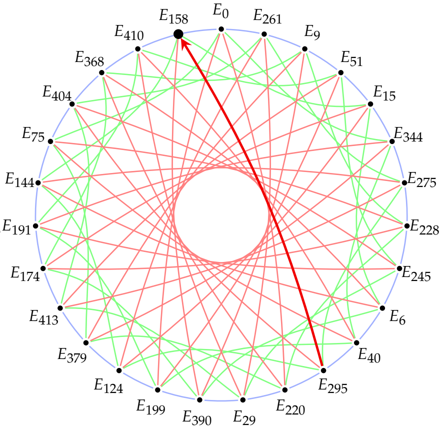


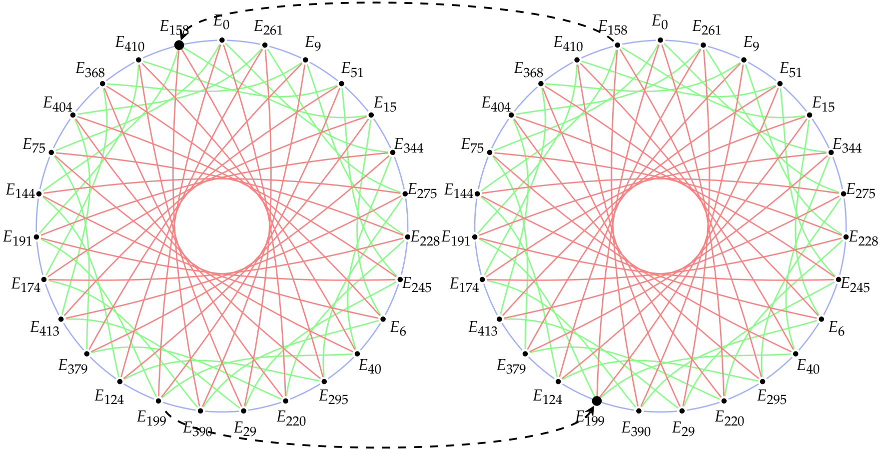
Diffie-Hellman on ‘nice’ graphs

Alice

$a = [+, -, +, -]$

Bob

$b = [+, +, -, +]$



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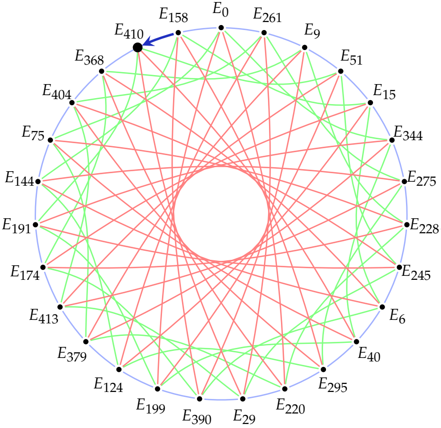

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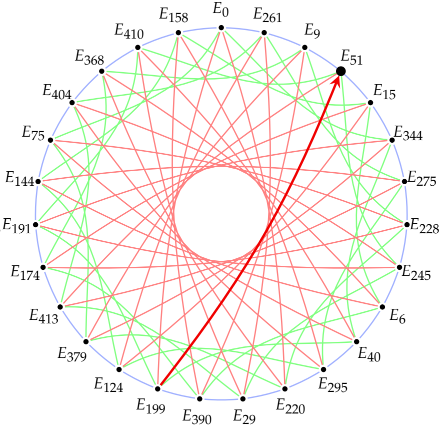

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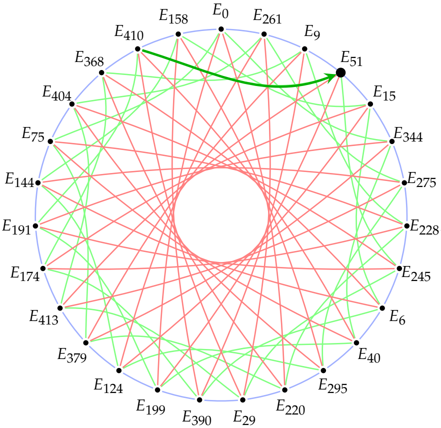

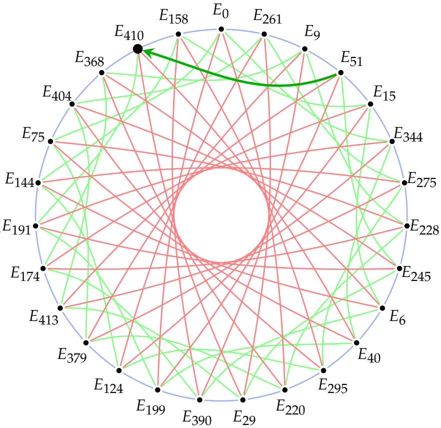

Diffie-Hellman on 'nice' graphs

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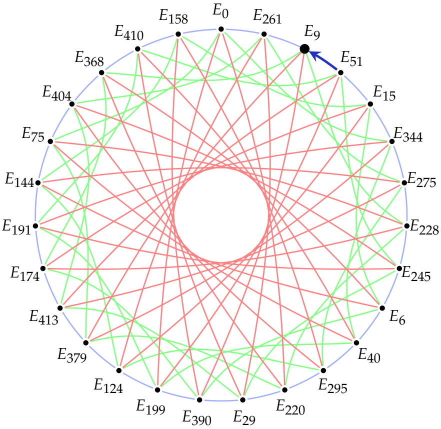

Bob

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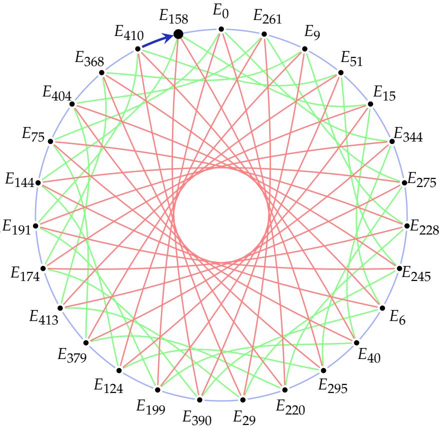

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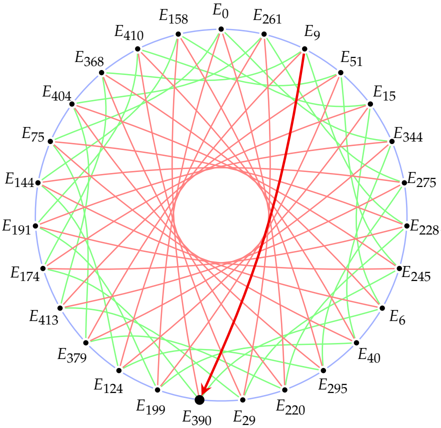

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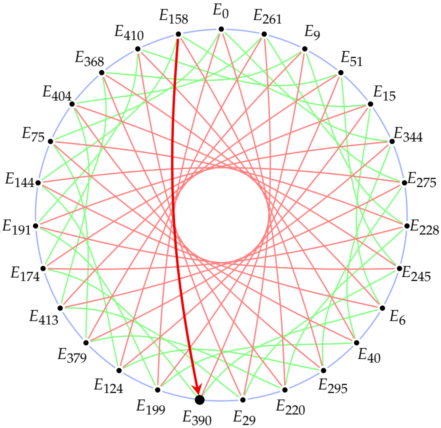

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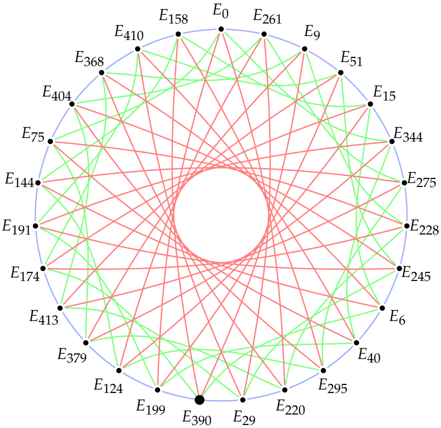

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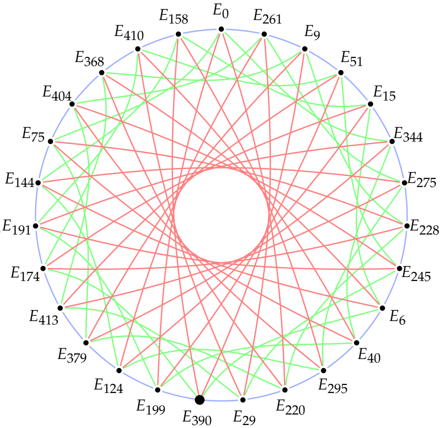

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- There is a geometric group law called $+$ on the rational points of E.
- The point at infinity P_∞ is the identity of the group.

The group of rational points on E is

$$E(\mathbb{F}_p) = \{(x, y) \in \mathbb{F}_p^2 : y^2 = f(x)\} \cup \{P_\infty\}.$$

Example

Define $E/\mathbb{F}_5 : y^2 = x^3 + 1$. Then

$$E(\mathbb{F}_5) = \{(0, 1), (0, -1), (2, 3), (2, -3), (-1, 0), P_\infty\}.$$

Elliptic curves

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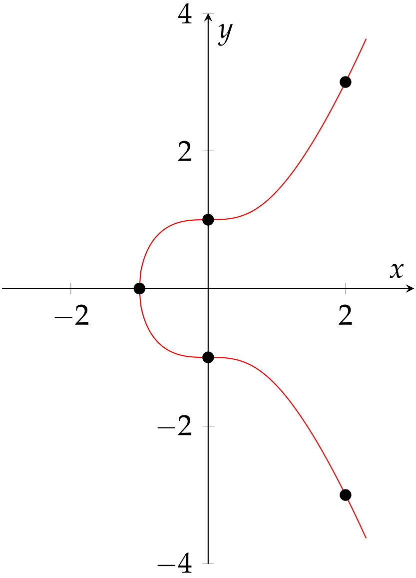

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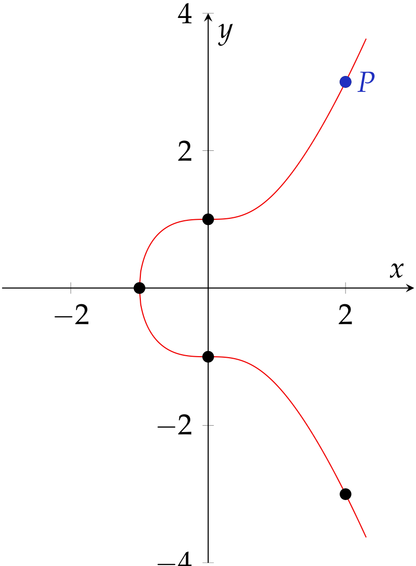

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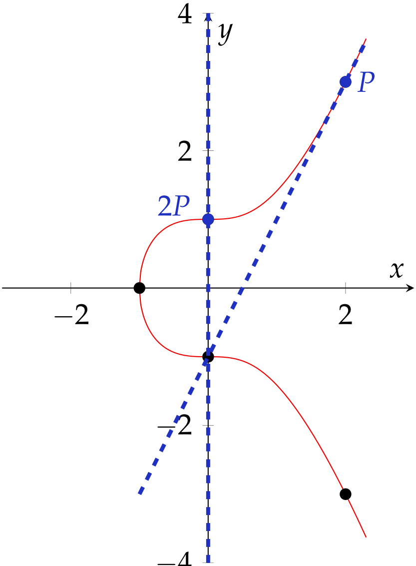

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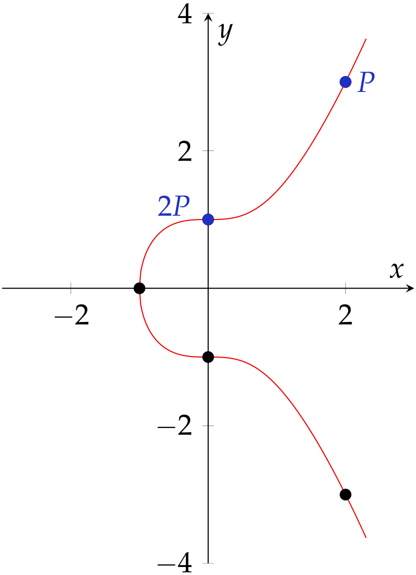

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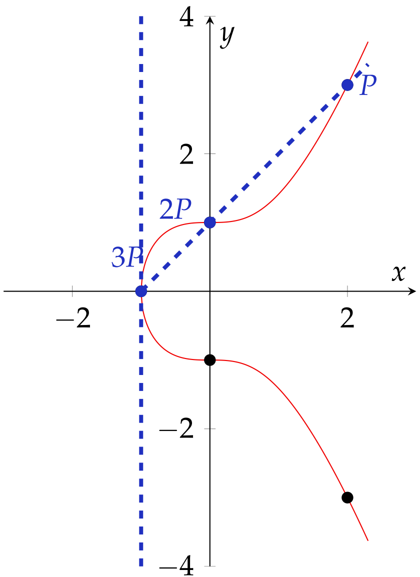

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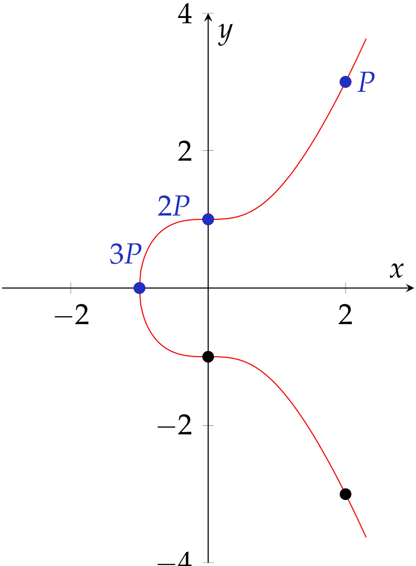

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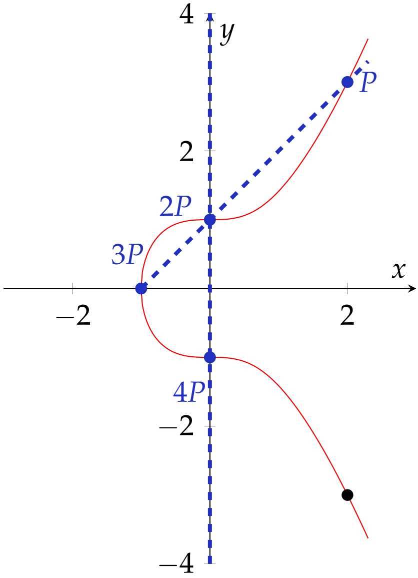

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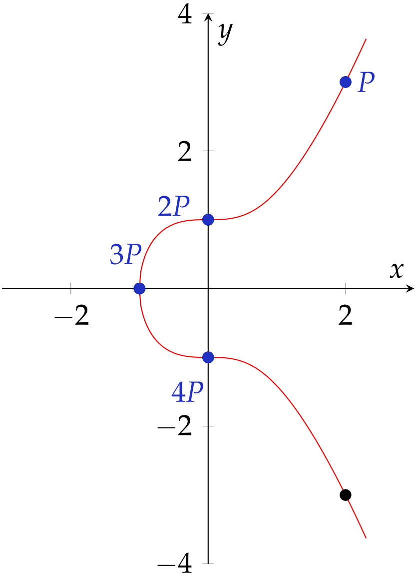

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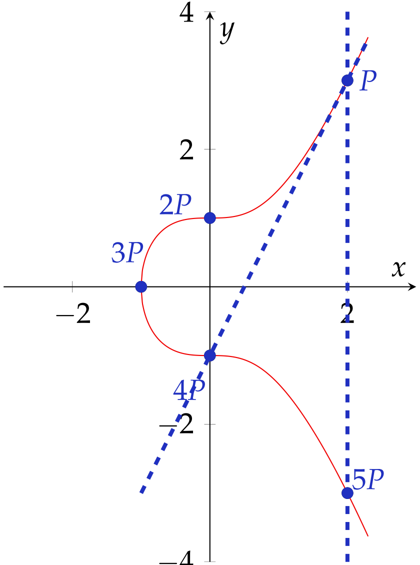

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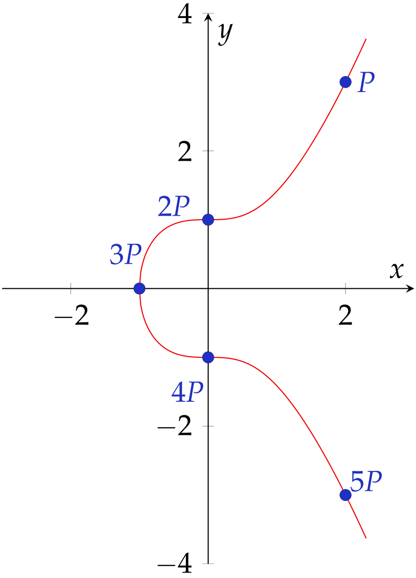

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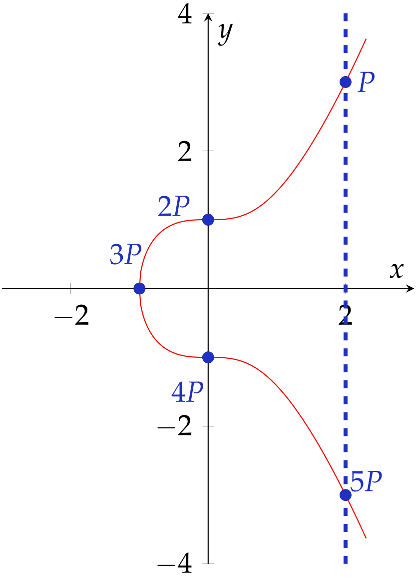

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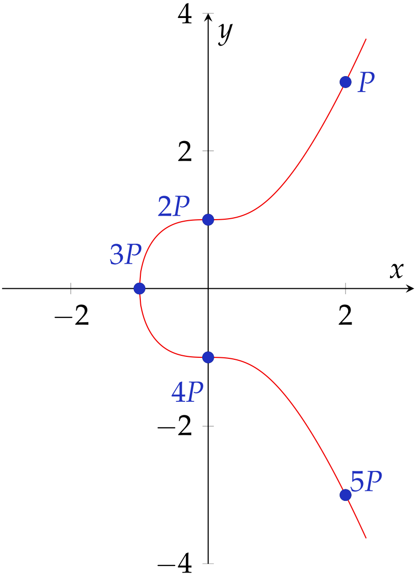
- $E(\mathbb{F}_5)$ is cyclic – $E(\mathbb{F}_5) \cong C_6$.



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Theorem

If $E/\mathbb{F}_p$ is supersingular and $p \geq 5$ then

$$E(\mathbb{F}_p) \cong C_{p+1} \quad \textit{or} \quad E(\mathbb{F}_p) \cong C_2 \times C_{(p+1)/2}.$$

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A point $P \in E(\mathbb{F}_p)$ is called a *n-torsion point* if $nP = P_\infty$.

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A point $P \in E(\mathbb{F}_p)$ is called a n-torsion point if $nP = P_\infty$. An n-torsion point P is a point of order n if there is no positive $m < n$ such that $mP = P_\infty$.

Example

$E/\mathbb{F}_5 : y^2 = x^3 + 1$. Then $E(\mathbb{F}_p) \cong C_6$ and is generated by $P = (2, 3)$.

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- $(2, 3)$ is a 6-torsion point of order 6.
- $(-1, 0) = 3(2, 3)$ is a 6-torsion point and a 2-torsion point, and has order 2.

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In either case, if $\ell | (p+1)$ is an odd prime, then $\frac{p+1}{\ell} P$ is a point of order ℓ.

Elliptic curves and isogenies

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An isogeny of elliptic curves over $\mathbb{F}_p$ is a non-zero morphism $E \to E'$ that maps the group identity of E to the group identity of E'. It is given by rational maps.

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Example

Define $E_{51}/\mathbb{F}_{419} : y^2 = x^3 + 51x^2 + x$

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\begin{array}{rccc}
[2]: & E_{51} & \to & E_{51} \\
 & (x, y) & \mapsto & 2 \cdot (x, y) := (x, y) + (x, y)
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- As $[2]$ is a morphism, it induces a morphism of groups $E(\mathbb{F}_{419}) \to E(\mathbb{F}_{419})$, i.e. $[2](P+Q) = [2](P) + [2](Q)$.

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- $[2](P_\infty) = P_\infty + P_\infty = P_\infty$. So $[2]$ maps the group identity of E_{51} to the group identity of E_{51}.

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Example

- Exercise: show that

$$\begin{array}{cccc} [2]: & E_{51} & \to & E_{51} \\ & (x,y) & \mapsto & \left(\frac{\frac{1}{2}x^4 - 18x^3 - 163x^2 - 18x + \frac{1}{2}}{8x(x^2+9x+1)}, \right. \\ & & & \left. \frac{y(x^6+18x^5+5x^4-5x^2-18x-1)}{(8x(x^2+9x+1))^2} \right). \end{array}$$

  Hint: Try to compute the rational maps using the group law from Mehdi's talk or see David's talk to learn how to compute the rational maps with Sage.

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Example

Fact: let $E_{51}/\mathbb{F}_{419} : y^2 = x^3 + 51x^2 + x$ and $E_9/\mathbb{F}_{419} : y^2 = x^3 + 9x^2 + x$ be elliptic curves. Then

$$\begin{array}{rccl} f: & E_{51} & \to & E_9 \\ & (x, y) & \mapsto & \left(\frac{x^3 - 183x^2 + 73x + 30}{(x+118)^2}, \right. \\ & & & \left. y \frac{x^3 - 65x^2 - 104x + 174}{(x+118)^3} \right). \end{array}$$

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The kernel $\ker(f)$ is the set of points (x, y) that map to the group identity P_∞:

- If $(x, y) \in \ker(f)$ then $(x, y) = P_\infty$ or $x = -118$.

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- If $(-118,y) \in E_{51}$ then $(x,y) = (-118, \pm 51)$.

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- If $(x, y) \in \ker(f)$ then $(x, y) = P_\infty$ or $x = -118$.
- If $(-118, y) \in E_{51}$ then $(x, y) = (-118, \pm 51)$.
- $f(P_\infty) = f((-118, \pm 51)) = P_\infty$.

Fact: an isogeny is uniquely determined by its kernel.

Elliptic curves and isogenies

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Definition

Let $E, E'/\mathbb{F}_p$ be elliptic curves and let ℓ be a prime different from p. An ℓ-isogeny $f : E \to E'$ is an isogeny with $\# \ker(f) = \ell$.

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$E_{51}/\mathbb{F}_{419} : y^2 = x^3 + 51x^2 + x$ and $E_9/\mathbb{F}_{419} : y^2 = x^3 + 9x^2 + x$.
The dual of the 3-isogeny $f : E_{51} \to E_9$ with kernel generated by $(-118, 51)$ is the 3-isogeny $f^\vee : E_9 \to E_{51}$ with kernel generated by $(286, 107\sqrt{380})$.

Isogeny graphs

Graph of 3-isogenies over $\mathbb{F}_{419}$.

Example

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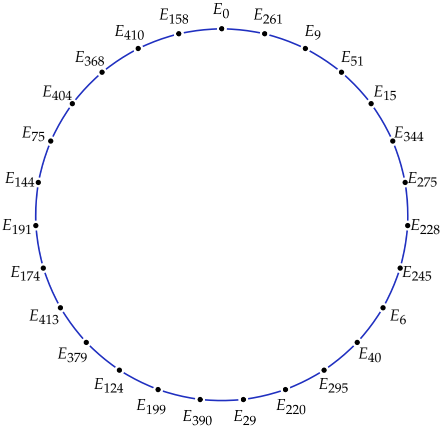
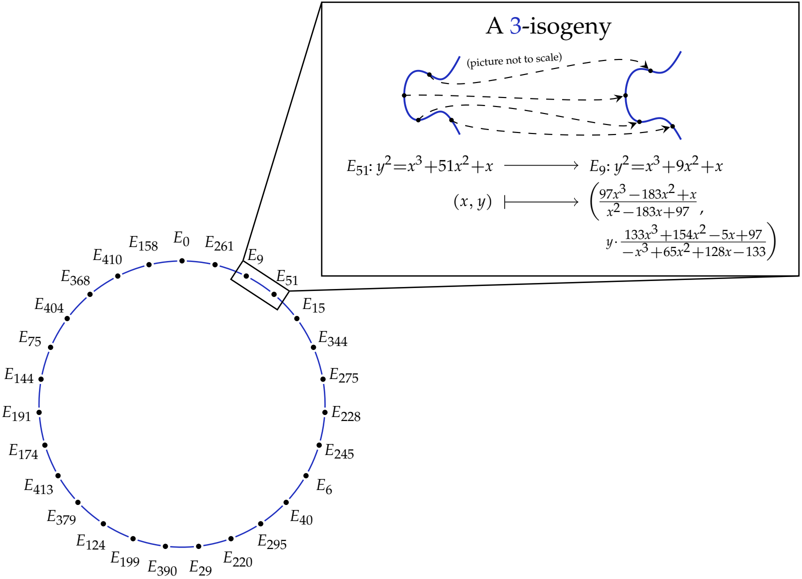


Isogeny graphs



Isogeny graphs

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- Nodes: elliptic curves defined over $\mathbb{F}_p$ with a given number of points (up to $\mathbb{F}_p$-isomorphism).
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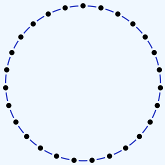

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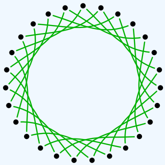

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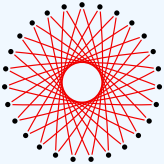

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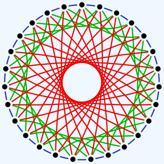
$G_3 \cup G_5 \cup G_7$



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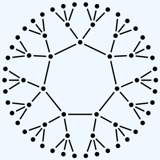
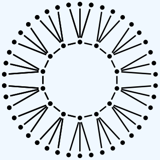
- Generally, the G_ℓ look something like

G_3: 

G_5: 

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Let $E/\mathbb{F}_p$ be supersingular.

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- Fact: if $p \equiv 3 \pmod 8$, $p \geq 5$, and $E_A/\mathbb{F}_p : y^2 = x^3 + Ax^2 + x$ is supersingular, then $\mathrm{End}_{\mathbb{F}_p}(E) \cong \mathbb{Z}[\sqrt{-p}]$.

Group actions

Remember: we wanted to replace exponentiation

$$\begin{array}{ccc} \mathbb{Z} \times G & \to & G \\ (x, g) & \mapsto & g^x := \underbrace{g * \cdots * g}_{x \text{ times}}. \end{array}$$

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Now we can do it!

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An action of a group $(H, \cdot)$ on a set S is a map

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such that $\mathrm{id} * s = s$ and $h_1 * (h_2 * s) = (h_1 \cdot h_2) * s$ for all $s \in S$ and all $h_1, h_2 \in H$.

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Example

Traditional Diffie-Hellman is an example:
$(H, \cdot) = ((\mathbb{Z}/(p-1)\mathbb{Z})^*, +)$ and $S = (\mathbb{Z}/p\mathbb{Z})^*$. Exponentiation $(h, s) \mapsto s^h$ is a group action.

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For the CSIDH group action

- the set S is the set of supersingular $E_A/\mathbb{F}_p : y^2 = x^3 + Ax^2 + x$ with $p \equiv 3 \pmod 8$ and $p \geq 5$.
- the group H is the class group of the endomorphism ring $\mathbb{Z}[\sqrt{-p}]$.

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Let $\mathcal{O} = \mathbb{Z}[\sqrt{-p}]$.

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An ideal $I \subset \mathcal{O}$ is the set of all $\mathcal{O}$-linear combinations of a given set of elements of $\mathcal{O}$.

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- We can multiply ideals I and $J \subset \mathcal{O}$:

$$I \cdot J = \langle \alpha\beta : \alpha \in I, \beta \in J \rangle.$$

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Two ideals $I, J \subseteq \mathcal{O}$ are equivalent if there exist $\alpha, \beta \in \mathcal{O} \setminus \{0\}$ such that

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[2] modulo details

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Miracle fact: the ideal class group is a group!

[2] modulo details

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The class group of the endomorphism ring $\mathbb{Z}[\sqrt{-p}]$ acts on the set S of supersingular elliptic curves $E_A/\mathbb{F}_p : y^2 = x^3 + Ax^2 + x$ with $p \equiv 3 \pmod 8$ and $p \geq 5$.

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- The isogeny f_I is an ℓ-isogeny if and only if $I = \langle [\ell], \pi \pm [1] \rangle$.

Class group action

The CSIDH group action is:

$$\begin{array}{ccc} \mathrm{Cl}(\mathrm{End}_{\mathbb{F}_p}(E)) \times S & \to & S \\ (I, E) & \mapsto & I * E := f_I(E). \end{array}$$

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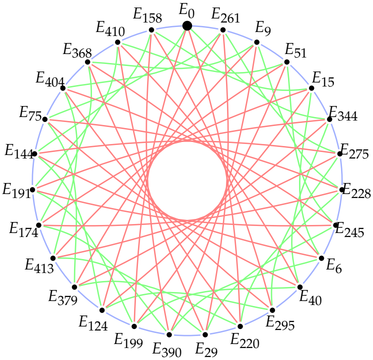
- A $'-'$ direction isogeny on the ℓ-isogeny graph is the action of $\langle [\ell], \pi + [1] \rangle$.

Diffie-Hellman with CSIDH

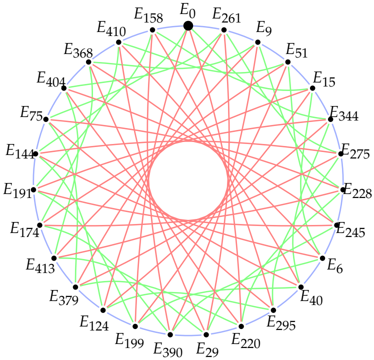
Alice

$a = [+, -, +, -]$



Bob

$b = [+, +, -, +]$



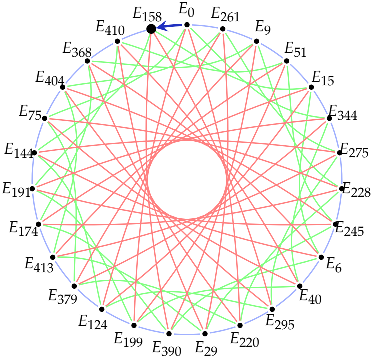
Diffie-Hellman with CSIDH

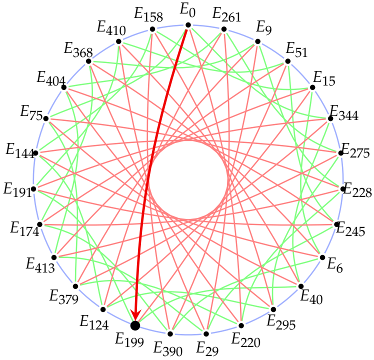
Alice

$a = [+, -, +, -]$
$\quad\uparrow$

Bob

$b = [+, +, -, +]$
$\quad\uparrow$





$$E_{158} = \langle 3, \pi - 1 \rangle * E_0 \quad E_{199} = \langle 7, \pi - 1 \rangle * E_0$$

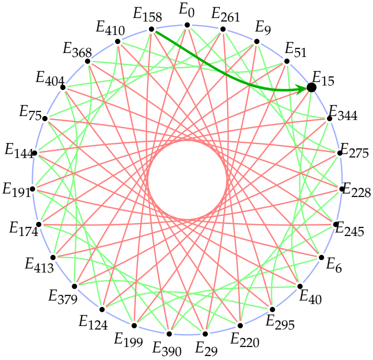
Diffie-Hellman with CSIDH

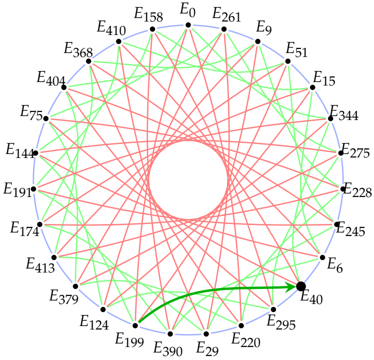
Alice

$$a = [+, \underset{\uparrow}{-}, +, -]$$

Bob

$$b = [+, \underset{\uparrow}{+}, -, +]$$





$$E_{15} = \langle 5, \pi + 1 \rangle * E_{158} \quad E_{40} = \langle 5, \pi - 1 \rangle * E_{199}$$

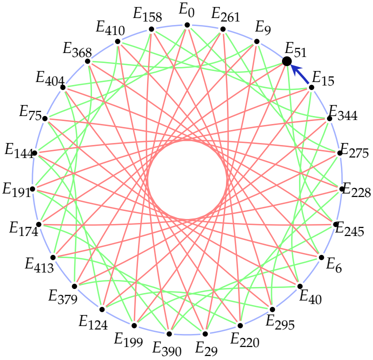
Diffie-Hellman with CSIDH

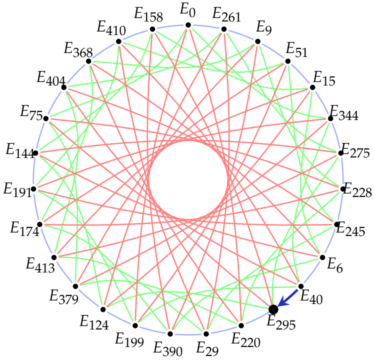
Alice

$$a = [+, -, \underset{\uparrow}{+}, -]$$

Bob

$$b = [+, +, \underset{\uparrow}{-}, +]$$



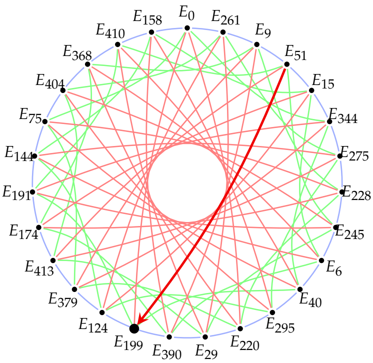


$$E_{15} = \langle 3, \pi - 1 \rangle * E_{51} \quad E_{295} = \langle 3, \pi + 1 \rangle * E_{40}$$

Diffie-Hellman with CSIDH

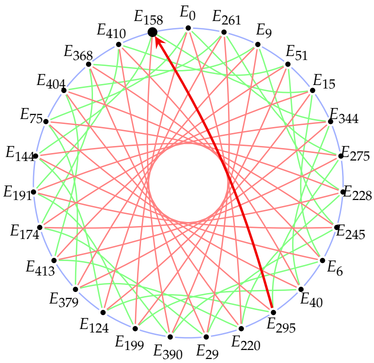
Alice

$$a = [\color{blue}{+}, \color{green}{-}, \color{blue}{+}, \color{red}{-}]$$
$$\uparrow$$



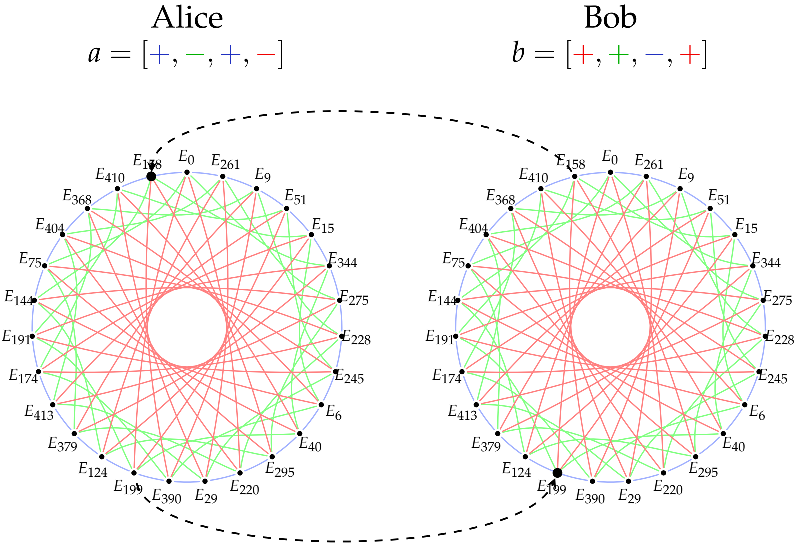
Bob

$$b = [\color{red}{+}, \color{green}{+}, \color{blue}{-}, \color{red}{+}]$$
$$\uparrow$$



$$E_{199} = \langle \color{red}{7}, \pi + 1 \rangle * E_{51} \quad E_{158} = \langle \color{red}{7}, \pi - 1 \rangle * E_{295}$$

Diffie-Hellman with CSIDH



(exchange of public keys)

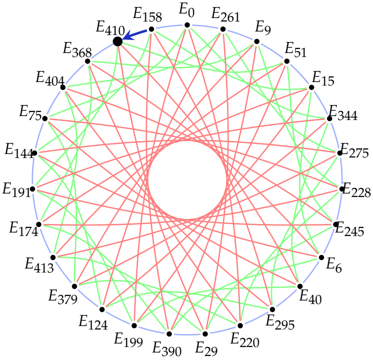
Diffie-Hellman with CSIDH

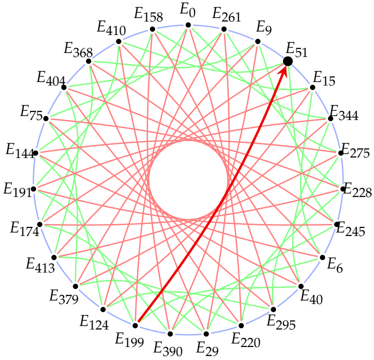
Alice

$a = [+, -, +, -]$
$\uparrow$

Bob

$b = [+, +, -, +]$
$\uparrow$

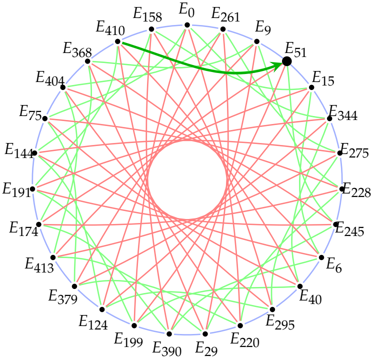


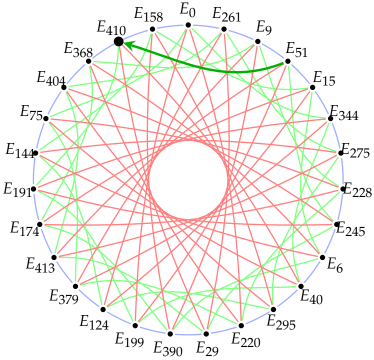


$$E_{410} = \langle 3, \pi - 1 \rangle * E_{158} \quad E_{51} = \langle 7, \pi - 1 \rangle * E_{199}$$

Diffie-Hellman with CSIDH

Alice	Bob
$a = [+, -, +, -]$	$b = [+, +, -, +]$
$\uparrow$	$\uparrow$





$$E_{51} = \langle 5, \pi + 1 \rangle * E_{410} \quad E_{410} = \langle 5, \pi - 1 \rangle * E_{51}$$

Diffie-Hellman with CSIDH

Alice

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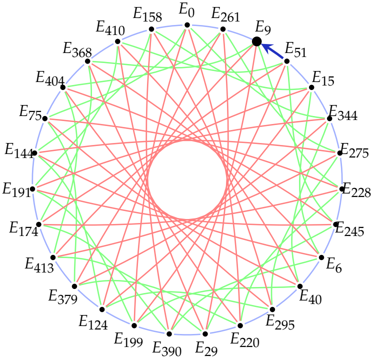
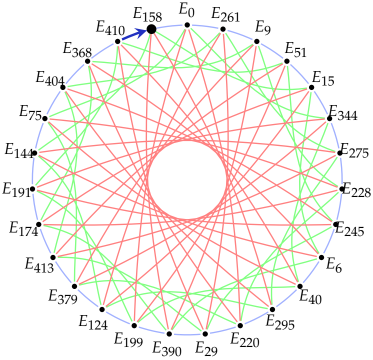
$\uparrow$



Bob

$b = [+, +, -, +]$
$\uparrow$



$$E_9 = \langle 3, \pi - 1 \rangle * E_{51} \quad E_{158} = \langle 3, \pi + 1 \rangle * E_{410}$$

Diffie-Hellman with CSIDH

Alice

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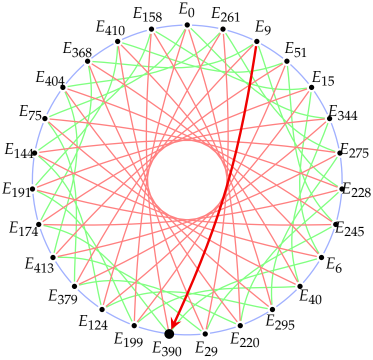
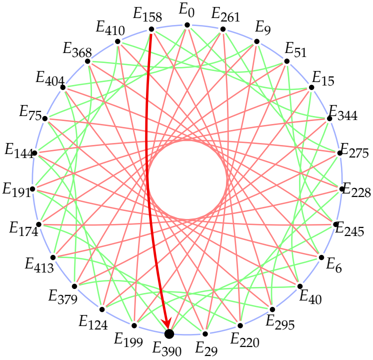


Bob

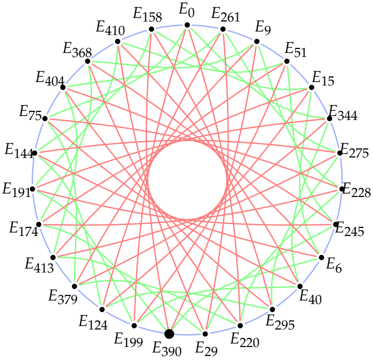
$b = [+, +, -, \underset{\uparrow}{+}]$

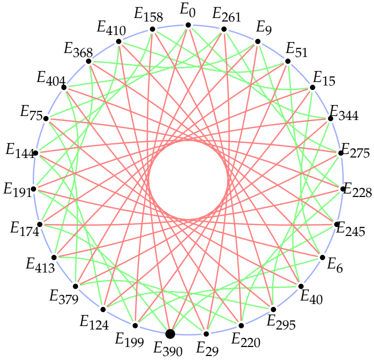


$$E_{390} = \langle 7, \pi + 1 \rangle * E_9 \quad E_{390} = \langle 7, \pi - 1 \rangle * E_{158}$$

Diffie-Hellman with CSIDH

Alice	Bob
$a = [+, -, +, -]$	$b = [+, +, -, +]$





(shared secret key is E_{390})

Design choices

- Choose small odd primes $\ell_1, \ldots, \ell_n$.

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- All arithmetic for computing ℓ_i-isogenies is now over $\mathbb{F}_p$. (For more: see David's talk).
- Every G_{ℓ_i} containing E_0 is a disjoint union of cycles.
- Every node of G_{ℓ_i} is of the form $E_A : y^2 = x^3 + Ax^2 + x$ – can be compressed to just $A \in \mathbb{F}_p$ giving tiny keys.

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- Small keys: 64 bytes at conjectured AES-128 security level
- Competitive speed: $\sim 85\,\text{ms}$ for a full key exchange
- Flexible: compatible with 0-RTT protocols such as QUIC; recent preprint uses CSIDH for 'SeaSign' signatures

Work in progress & future work

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- More applications.
- [Your paper here!]


Thank you!

References

Mentioned in this talk:

- Castryck, Lange, Martindale, Panny, Renes:
  CSIDH: An Efficient Post-Quantum Commutative Group Action
  `https://ia.cr/2018/383` (to appear at ASIACRYPT 2018)
- [BLMP] Bernstein, Lange, Martindale, Panny:
  Quantum circuits for the CSIDH: optimizing quantum evaluation of isogenies
  `https://eprint.iacr.org/2018/1059`
- De Feo, Galbraith:
  SeaSign: Compact isogeny signatures from class group actions
  `https://ia.cr/2018/824`

Credits should also go to Lorenz Panny - many of the slides from this presentation are from a joint presentation with Lorenz at the Crypto Working Group in Utrecht, the Netherlands. He made all the beautiful pictures! Also credits to Wouter Castryck, whose slides were a source of inspiration for this presentation.

References

Other related work:

- Biasse, Iezzi, Jacobson:
  A note on the security of CSIDH
  `https://arxiv.org/pdf/1806.03656` (to appear at Indocrypt 2018)
- Bonnetain, Schrottenloher:
  Quantum Security Analysis of CSIDH and Ordinary Isogeny-based Schemes[3]
  `https://ia.cr/2018/537`
- Childs, Jao, Soukharev:
  Constructing elliptic curve isogenies in quantum subexponential time
  `https://arxiv.org/abs/1012.4019`
- Delfs, Galbraith:
  Computing isogenies between supersingular elliptic curves over $\mathbb{F}_p$
  `https://arxiv.org/abs/1310.7789`
- De Feo, Kieffer, Smith:
  Towards practical key exchange from ordinary isogeny graphs
  `https://ia.cr/2018/485` (to appear at ASIACRYPT 2018)
- Jao, LeGrow, Leonardi, Ruiz-Lopez:
  A polynomial quantum space attack on CRS and CSIDH
  (to appear at MathCrypt 2018)
- Meyer, Reith:
  A faster way to the CSIDH
  `https://ia.cr/2018/782` (to appear at Indocrypt 2018)

[3] Concrete numbers in this paper should be treated with caution, see [Section 1.3, BLMP]

Parameters

CSIDH-log p	intended NIST level	public key size	private key size	time (full exchange)	cycles (full exchange)	stack memory	classical security
CSIDH-512	1	64 b	32 b	85 ms	212e6	4368 b	128
CSIDH-1024	3	128 b	64 b				256
CSIDH-1792	5	224 b	112 b				448